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Strong Law of large number
Law of the iterated logarithm
for nonlinear probabilities

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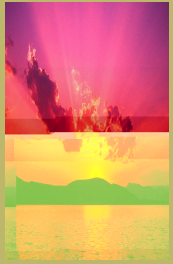
Outline

◇ *History of LLN and LIL for probabilities*

◇ *Why to study LLN and LIL for capacities*

◇ *Nonlinear probabilities and nonlinear expectations*

◇ *Main results*

◇ *Applications*

0.1. History of LLN and LIL for probability

Law of large number(LLN):

(1) Brahmagupta (598-668), Cardano (1501-1576)

(2) Jakob Bernoulli(1713), Poisson (1835)

(3) Chebyshev, Markov, Borel(1909), Cantelli and Kolmogorov(IID).

Law of iterated logarithm(LIL):

(1) Khintchine(1924) for Bernoulli model

Kolmogorov(1929), Hartman–Wintner(1941) (IID)

(2) Levy(1937) for Martingale

(3) Strassen(1964) for functional random variables.

0.2. Strong LLN and LIL for probabilities

Assumption: $\{X_i\}$ IID , $S_n/n := \sum_{i=1}^{n} X_i$, $EX_1 = \mu$, Then

Theorem 1: Kolmogorov:

$$P(\lim_{n\to\infty} S_n/n = \mu) = 1$$

Theorem 2: Hartman–Wintner(1941): If $EX_1 = 0, EX_1^2 = {}^2$, Then

(a)

$$P\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = \right) = 1$$

(b)

$$P\left(\liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = - \right) = 1$$

(c) Suppose that $C(\{x_n\})$ is the cluster set of a sequence of $\{x_n\}$ in R, then

$$P\left(C(\{ \ : S_n(\ )/\sqrt{2n\log\log n}\}) = [- \ , \ ]\right) = 1.$$

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0.3. Why to study LLN and LIL in Finance

THEOREM 1 *(Black-Scholes, 1973:) In complete markets, there exists a unique probability measure Q, such that the pricing of option at strike date T is given by $E_Q[\ e^{-rT}]$. Where $r = 0$ is interest rate of bond.*

Monte Carlo, $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E_Q[\ ]$.

(Linear) expectation $\leftarrow$ Black-Scholes $\rightarrow$ Complete Markets

$\inf_{Q\in\mathcal{P}} E_Q[\ ], \sup_{Q\in\mathcal{P}} E_Q[\ ] \iff$ Incomplete Markets, Q is not unique, SET $\mathcal{P}$.

Super-pricing: $\inf_{Q\in\mathcal{P}} E_Q[\ ], \sup_{Q\in\mathcal{P}} E_Q[\ ]$. Nonlinear expectation! $\lim_{n\to\infty} S_n/n =?$

0.4. **Bernoulli Trials with ambiguity**

Bernoulli Trials:

Repeated independent trials are called Bernoulli trials if there are only two possible outcomes for each trial and their probabilities REMAIN (are no longer) the same throughout the trials.

Let $X_i = 1$ if head occurs and $X_i = 0$ if tail occurs.

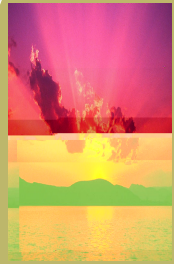$$P(X_i = 1) = \quad , \quad P(X_i = 0) = 1 - \quad , \quad S_n := \sum_{i=1}^{n} X_i$$

If $\quad = 1/2$ (Unbalance), LLN stats

$$P(\lim_{n \to \infty} S_n/n = 1/2) = 1$$

Or

$$\lim_{n \to \infty} S_n/n = 1/2 \quad a.s \quad (P)$$

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If a coin is balance. $P(X_i = 1) = \quad \in [1/3, 1/2]$.

Let $\mathcal{P} := \{P, \quad \in [1/3, 1/2]\}$.

$E_P[X_i] = \quad$ Unknown,

But $\max_{P \in \mathcal{P}} E_P[X_i] = 1/2, \quad \min_{P \in \mathcal{P}} E_P[X_i] = 1/3$.

Question: what is the limit $S_n/n \to$?

(a) Capacity: If $V(A) := \max_{P \in \mathcal{P}} P(A), \quad v(A) := \min_{P \in \mathcal{P}} P(A)$

Can S_n/n converge to $\max_{P \in \mathcal{P}} E_P[X_i]$ or $\min_{P \in \mathcal{P}} E_P[X_i]$ a.s. V or v?

(b) The relation between the set of limit points of S_n/n and the interval of $\min_{P \in \mathcal{P}} E_P[X_i]$ and $\max_{P \in \mathcal{P}} E_P[X_i]$.

0.5. Linear and Nonlinear Expectations

Kolmogorov: Linear expectation:$P : \mathcal{F} \to [0, 1], P(A) = E[I_A]$

$$P(A + B) = P(A) + P(B), \ A \cap B = \emptyset \Leftrightarrow E[\ + \ ] = E[\ ] + E[\ ]$$

Expectation is a linear functional of random variable.

Nonlinear probability(capacity): $V(\cdot) : \mathcal{F} \to [0, 1]$ but

$$V(A + B) \neq V(A) + V(B), \text{even} A \cap B = \emptyset.$$

Nonlinear expectation: $\mathbb{E}(\ )$ is nonlinear functional in the sense of

$$\mathbb{E}[\ + \ ] \neq \mathbb{E}[\ ] + \mathbb{E}[\ ].$$

Capacity $V(A) = \mathbb{E}[I_A]$ is nonlinear.

Modes of nonlinear expectations and capacity

(1)Choquet expectations (Choquet 1953, physics)

$$C_V[X] := \int_0^\infty V(X \geq t)dt + \int_{-\infty}^0 [V(X \geq t) - 1]dt.$$

(2)g-expectation (Peng 1997)

(3) Sub-linear expectation(Peng 2007).

(a)Monotonicity: $X \geq Y$ implies $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

(b)Constant preserving: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$.

(c)Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.

(d)Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.

(1) Distorted probability measure: $V(A) = g(P(A)), g : [0, 1] \to [0, 1]$.

(2) 2-alternating capacity: $V(A \cup B) \leq V(A) + V(B) - V(A \cap B)$

(3) $V(A) = \max_{P \in \mathcal{P}} P(A), \mathcal{P}$ set of Probability.

3. Definition: capacity and nonlinear expectation

(1) Probability space :$(\ ,\mathcal{F},P) \Rightarrow (\ ,\mathcal{F},\mathcal{P})$. Where $\mathcal{P} := \{P : \ \in \ \}$.

(2) Capacity: $P \Rightarrow (v, V)$, where

$$v(A) = \inf_{Q \in \mathcal{P}} Q(A), \quad V(A) = \sup_{Q \in \mathcal{P}} Q(A).$$

(3) Property:

$$V(A) + V(A^c) \geq 1, \quad v(A) + v(A^c) \leq 1$$

but

$$V(A) + v(A^c) = 1.$$

(4) Nonlinear expectations: Lower-upper expectation $\mathcal{E}[\ ]$ and $\mathbb{E}[\ ]$

$$\mathcal{E}[\ ] = \inf_{Q \in \mathcal{P}} E_Q[\ ], \quad \mathbb{E}[\ ] = \sup_{Q \in \mathcal{P}} E_Q[\ ]$$

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$$V(AB) = V(A)V(B), v(AB) = v(A)v(B)$$

Theorem (Epstein, 02, Marinacci, 99, 05).   Bounded,   Polish, $C_v[X_i] = \underline{\mu}$, $C_V[X_i] = \overline{\mu}$. $\{X_i\}$ IID, then

$$v\left(\underline{\mu} \le \liminf_{n\to\infty} S_n/n \le \limsup_{n\to\infty} S_n/n \le \overline{\mu}\}\right) = 1.$$

Where V is totally 2-alternating $V(A \bigcup B) \le V(A) + V(B) - V(AB)$, here C_v and C_V is Choquet are integrals.

Note $C_v[X] \le \mathcal{E}[X] \le \mathbb{E}[X] \le C_V[X], \forall X$.

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4.1. Limit theorem 1

Theorem: If $\{X_i\}$ is IID, then $\frac{S_n}{n}$ converges as $n \to \infty$ a.s. ν if and only if

$$\mathcal{E}[X_1] = \mathbb{E}[X_1].$$

In this case,

$$\lim S_n/n = \mathcal{E}[X_1], \quad a.s. \quad \nu.$$

5. Main results

THEOREM 3 $\{X_i\}_{i=1}^n$ *IID under nonlinear expectation* $\mathbb{E}$. *Set* $\overline{\mu} := \mathbb{E}[X_i]$, $\underline{\mu} := \mathcal{E}[X_i]$ *and* $S_n := \sum_{i=1}^n X_i$. *If* $\mathbb{E}[|X_i|^{1+}] < \infty$ *for* > 0. *Then*

(I)

$$v\left(\in : \underline{\mu} \le \liminf_{n\to\infty} S_n(\ )/n \le \limsup_{n\to\infty} S_n/n(\ ) \le \overline{\mu}\}\right) = 1.$$

(II)

$$V\left(\in : \limsup_{n\to\infty} S_n(\ )/n = \overline{\mu}\right) = 1$$

$$V\left(\in : \liminf_{n\to\infty} S_n(\ )/n = \underline{\mu}\right) = 1.$$

(III) Suppose that $C(\{S_n(\ )/n\})$ *is the cluster set of a sequence of* $\{S_n(\ )/n\}$, *then*

$$V\left(\in : C(\{S_n(\ )/n\}) = [\underline{\mu}, \overline{\mu}]\right) = 1$$

6. Law of iterated logarithm for sub-linear expectations

THEOREM 4 $\{X_n\}$ *bounded IID.* $\mathbb{E}[X_1] = \mathcal{E}[X_1] = 0$, $\overline{}^{-2} := \mathbb{E}[X_1^2]$, $\underline{}^2 := \mathcal{E}[X_1^2]$. *Let* $S_n := \sum_{i=1}^{n} X_i$, $a_n := \sqrt{2n \lg \lg n}$, *then*

(I)

$$v\left(\underline{} \leq \limsup_{n} \frac{S_n}{a_n} \leq \overline{}\right) = 1;$$

(II)

$$v\left(-\overline{} \leq \liminf_{n} \frac{S_n}{a_n} \leq -\underline{}\right) = 1.$$

(III) *Suppose that* $C(\{x_n\})$ *is the cluster set of a sequence of* $\{x_n\}$ *in* R, *then*

$$v\left(C(\{S_n / \sqrt{2n \log\log n}\}) \supset (-\underline{}, \underline{})\right) = 1.$$

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7. Key of proof

THEOREM 5 *Suppose is distributed to G normal $N(0; [\_^2, ^{-2}])$, where $0 < \_ \leq ^{-} < \infty$. Let be a bounded continuous function. Furthermore, if is a positively even function, then, for any $b \in R$,*

$$e^{-\frac{b^2}{2\_^2}} \mathcal{E}[\ (\ )] \leq \mathcal{E}[\ (\ - b)].$$

8.   Application

Total 100 balls in box, Black + Red + Yellow = 100,

Black = Red, Yellow $\in [30, 40]$, then $P_Y \in [3/10, 4/10]$.

Take a ball from this box,

$X_i = 1$, if ball is black, $X_i = 0$, if ball is Yellow, $X_i = -1$ for red.

$S_n = \sum_{i=1}^{n} X_i$, is the excess frequency of black than Red

Then

(a) $\mathbb{E}[X_i] = \mathcal{E}[X_i] = 0$

(b)

$$\sqrt{6/10} \leq \limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \lg \lg n}} \leq \sqrt{7/10}.$$

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